1-D non-linear inversion of data with absorption - revisited

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Summary

We revisit the classical problem of 1-D non-linear inversion of seismic data with absorption. The basic formulation of the forward problem is governed by the Riccati equation and with interbed multiples included (Gjevik et al, 1976). In this paper, the corresponding nonlinear inversion is solved by the use of a least-squares formulation in the frequency-domain with regularization included. A simple synthetic data example is employed to demonstrate the approach.

Non-linear modeling of absorption employing the Riccati equation

Assume monochromatic plane-waves propagating along a vertical axis. Let P define the stress (pressure) and W the displacement. Density is ρ. Newton's second law gives:

$$\frac{dP}{dz} + \rho\omega^2 W = 0 \tag{1}$$

Correspondingly, a stress-strain relationship of the following form is assumed (Hook's law):

$$P = \rho \upsilon_r^2 Y \frac{dW}{dz} \tag{2}$$

In Eq.(2) v_r is the reference velocity which could be taken as the group velocity in case of dispersion. The function Y represents depth and frequency-dependent absorption.

In case of no damping, Y=1 and Eq.(2) is simply Hookes law.

Combination of Eqs.(1) and (2) gives Helmholtz equation (assume constant density)

$$\frac{d^2P}{dz^2} + k^2 P = 0 \ , k = \frac{\omega}{\upsilon_r \sqrt{Y}} \tag{3}$$

In case of a layer with constant velocity and absorption function Y, plane-wave solutions of Eq. (3) can be written formally on the simple form (positive z-axis pointing downwards)

U=a exp[ikz] , D=b exp[-ikz] , $k=k_r - k_i$ ($k_r, k_i \geq 0$) (4)

With U and D representing respectively upward and downward propagating components. Thus, the total field can be written

P=U+D (5)

From eq. (1) and by using Eqs.(4) and (5) it follows that

$$W = -\frac{1}{i\omega\rho v_r \sqrt{Y}}(D-U) \tag{6}$$

A depth-varying model can be assumed as the limit of an infinite number of infinitesimal layers. For such a model, the relation in Eq. (6) is also assumed to be valid. Differentiation of Eq.(6) with respect to z gives (also making use of Eq.(1))

$$\frac{dW}{dz} = \frac{1}{\rho v_r^2 Y} P = \frac{1}{\rho v_r^2 Y}(U+D) = -\frac{1}{i\omega\rho v_r \sqrt{Y}}\left(\frac{dD}{dz} - \frac{dU}{dz}\right) - \frac{1}{i\omega}\frac{d}{dz}\left(\frac{1}{\rho v_r \sqrt{Y}}\right)(D-U) \tag{7}$$

Consider now the last term on the RHS of Eq.(7), a further simplification can be obtained by studying (small to moderate absorption)

$$\frac{d}{dz}\left(\frac{1}{\rho v_r\sqrt{Y}}\right)=-\frac{1}{\rho v_r\sqrt{Y}}\left[\frac{1}{\sqrt{Y}}\frac{d\sqrt{Y}}{dz}+\frac{1}{\rho v_r,}\frac{d(\rho v_r)}{dz}\right]\cong-\frac{2}{\rho v_r\sqrt{Y}}r\,, \tag{8}$$

$$r(z)=\frac{i}{2\rho v_r}\frac{d(\rho v_r)}{dz}$$

Where r represents the depth-dependent 'reflectivity' per depth unity

Combination of Eqs.(1) and (6) gives

$$\frac{dP}{dz}=\frac{dU}{dz}+\frac{dD}{dz}=\frac{\omega}{iv_r\sqrt{Y}}(D-U) \tag{9}$$

A main result is obtained now by combining Eqs. (7)-(9)

$$\begin{aligned}\frac{dD}{dz}&=-\frac{i\omega}{v_r\sqrt{Y}}D+r\,(D-U)\\ \frac{dU}{dz}&=\frac{i\omega}{v_r\sqrt{Y}}U-r\,(D-U)\end{aligned} \tag{10}$$

Introduce now the ratio K=U/D that is the reflection coefficient, and differentiate it with respect to depth

$$\frac{dK}{dz}=\frac{d(U/D)}{dz}=\frac{1}{D}\frac{dU}{dz}-\frac{U}{D^2}\frac{dD}{dz} \tag{11}$$

Finally, by combining Eqs.(10) and (11) gives the Riccati equation (Gjevik et al, 1976)

$$\frac{dK}{dz}=\frac{2i\omega}{v_r\sqrt{Y}}K-r\,(1-K^2) \tag{12}$$

Since vertically travelling waves are considered, the transformation from depth to two-way traveltime is straightforward

$$\tau=2\int_0^z\frac{dz}{v_r},\ \Rightarrow d\tau=\frac{2}{v_r}dz \tag{13}$$

Which gives the travel time version of Eq.(29)

$$\frac{dK(\omega,\tau)}{d\tau}=\frac{i\omega}{\sqrt{Y(\omega,\tau)}}K(\omega,\tau)-r(\tau)(1-K^2),r(\tau)=\frac{i}{2\rho v_r}\frac{d(\rho v_r)}{d\tau} \tag{14}$$

By noticing that

$\exp(-i\omega\int_0^\tau Y(\omega,\tau))^{-1/2}d\tau)\equiv\exp\left[-\phi(\omega,\tau)\right]$ is an integrating factor for this Riccati equation, it can be rewritten on the following form:

$$\frac{d}{d\tau}\left[K(\omega,\tau)\exp(-\phi(\omega,\tau)\right] = -r(\tau)(1-K^2)\ \exp(-\phi(\omega,\tau) \tag{15}$$

Where

$$\phi(\omega,\tau) = i\omega\int_0^\tau \frac{d\tau}{\sqrt{Y(\omega,\tau)}} \tag{16}$$

Assume now the following boundary condition: K=0 when τ≥T. Integration of Eq. (15) now gives the solution

$$-K(\omega,\tau)\exp(-\phi(\omega,\tau)) = -\int_\tau^T r(\tau)\exp(-\phi(\omega,\tau'))(1-K^2(\omega,\tau'))\,d\tau' \tag{17}$$

Which can be arranged as

$$K(\omega,\tau) = \exp(\phi(\omega,\tau))\int_\tau^T r(\tau')\exp(-\phi(\omega,\tau'))(1-K^2(\omega,\tau'))\,d\tau' \tag{18}$$

Equation (18) is now the starting point for a modeling algorithm.

Choice of absorption model

Following Horton (1959) we introduce the notation for the absorption function Y

$$Y(\omega,\tau) = A(\omega,\tau) + iB(\omega,\tau) \tag{19}$$

In his paper, Horton gives examples of values of A and B for various absorption models that can be causal or non-causal. Employing Eq (3) the wavenumber *k* can now be expressed as (B<(<)A)

$$k = \frac{\omega}{v_r\sqrt{Y}} = \frac{\omega}{v_r\sqrt{A+iB}} = \frac{\omega}{v_r}\left[\frac{1}{\sqrt{A}} - \frac{i}{2}\frac{B}{A\sqrt{A}}\right] \tag{20}$$

Our objective is to find a set of A and B which fulfills the causality criterion. For a more complete discussion about dispersion and attenuation models, the reader is referred to Ursin and Toverud (2002). Sørsdal (1981) was the first to introduce Eq. (20) into the Riccati equation. In case of a constant-Q model (Kjartansson, 1979), the wavenumber *k* can be written on the following form

$$k = \frac{\omega}{v(\omega)}\left[1-\frac{i}{2Q}\right] = \frac{\omega}{v_r} + \left[\frac{\omega}{v(\omega)} - \frac{\omega}{v_r}\right] - i\alpha(\omega) = \frac{\omega}{v_r} + \varphi(\omega) - i\alpha(\omega).\ \ \alpha = \frac{\omega}{2Qv(\omega)} \tag{21}$$

Where α is the absorption coefficient and φ is the phase of the 'absorption filter'. In order to ensure causality, the filter should be minimum phase.

We introduce a Kolsky type of phase-velocity model (Kolsky, 1956)

$$v(\omega) = v_h \left(\frac{\omega}{\omega_h} \right)^{\gamma} \;,\; \gamma = (\pi Q)^{-1} \tag{22}$$

Which in combination with Eq.(21) gives

$$k = \frac{\omega}{v_h}\left[1 + \left[\left(\frac{\omega}{\omega_h} \right)^{-\gamma} - 1 \right] - \frac{i}{2Q}\left(\frac{\omega}{\omega_h} \right)^{-\gamma} \right] \tag{23}$$

Direct comparison between Eqs. (20) and (23) gives

$$A = \left[\frac{\omega}{\omega_r} \right]^{2\gamma} \quad B = \left[\frac{\omega}{\omega_r} \right]^{2\gamma} \frac{1}{Q} \tag{24}$$

In the actual application we follow Wang (2008) and choose the highest possible (tuning) frequency of the signal band as the reference. With time-sampling Δτ=0.004 s this will be the Nyquist frequency 125 Hz. Wang called this model 'modified Kolsky' This absorption model is assumed to be causal, at least in an approximate way, since it is closely related to the power law of Strick(1967) which satisfies the Kramers-Krönigs relations.

Non-linear inversion

Consider now Eq. (18) in the limit τ→0, which gives the 'seismogram'

$$K(\omega,0) = \int_0^T r(\tau')\exp(-\phi(\omega,\tau'))(1 - K^2(\omega,\tau'))\, d\tau' \tag{25}$$

Introduce 'reflectivity' series

$$r(\tau) = \Delta\tau \sum_{i=0}^{NT-1} r_i\, \delta(\tau - i\Delta\tau) \;, T = NT.\Delta\tau \tag{26}$$

Combination of Eqs. (25) and (26) gives

$$K(\omega,0) = \sum_{i=0}^{NT-1} r_i \exp(-\phi(\omega, i\Delta\tau))(1 - K^2(\omega, i\Delta\tau))\,\Delta\tau \tag{27}$$

Originally, seismogram recorded in timedomain, i.e. k(t,0), and assume sampled with a total of NT-samples. Fourier transform of the data will give the same number of monochromatic seismograms.

Hagos (2016) presented the theory in his thesis and made some computations based on the reflectivity. We have also made some calculations based on the impedance as Gjevik did in his paper. To do this we have to introduce the matrix system Eq.(28).

$$\begin{bmatrix} K_{n+1}(\omega_0,0) \\ K_{n+1}(\omega_1,0) \\ . \\ . \\ K_{n+1}(\omega_{NT-1},0) \end{bmatrix} = \qquad (28)$$

$$\begin{pmatrix} \exp(-\varphi(\omega_0,0)(1-K_{0,n}^2) & \exp(-\varphi(\omega_0,\Delta\tau)(1-K_{1,n}^2) & \dots & \exp(-\varphi(\omega_0,(NT-1)\Delta\tau)(1-K_{n,n}^2) \\ \exp(-\varphi(\omega_1,0)(1-K_{0,n}^2) & \exp(-\varphi(\omega_1,\Delta\tau)(1-K_{1,n}^2) & \dots & \exp(-\varphi(\omega_1,(NT-1)\Delta\tau)(1-K_{n,n}^2) \\ . & . & \dots & . \\ . & . & \dots & . \\ . & . & \dots & . \\ \exp(-\varphi(\omega_{NT-1},0)(1-K_{0,n}^2) & \exp(-\varphi(\omega_{NT-1},\Delta\tau)(1-K_{1,n}^2) & \dots & \exp(-\varphi(\omega_{NT-1},(NT-1)\Delta\tau)(1-K_{n,n}^2) \end{pmatrix} \begin{pmatrix} r_{n,0} \\ r_{n,1} \\ \vdots \\ r_{n,NT-1} \end{pmatrix}$$

For a given iteration number n, we can solve for the corresponding reflectivity series by solving Eq. (28) employing standard least squares inversion. Note that $K_0^2=0$. After a new estimate of the reflectivity series has been obtained, an update of $K_{i,n}{}^2$ can be obtained by solving the forward problem in Eq. (18). Iterations are carried out until the relative change in reflectivity is below a certain user threshold.

Numerical example

First controlled data was generated for the five-layer model described in Table 1 with absorption and interbed multiples. We use a Ricker pulse with center-frequency 40 Hz convolved with the reflectivity series Eq.(26). Figure 1.a. shows (left) the synthetics Eq. (18) with and without absorption and the inversion Eq.(28) (1.iteration) right. Fig.1.b. (left) shows the inversion with absorption after 5 iterations. The reflection coefficients of the model is to the right. The effects of absorption have been well compensated for, interbed multiples are removed and transmission loss restored. Also a small phase displacement for inversion with absorption is corrected for. This is visible for late arrivals on fig.1.a.(left) where synthetics with absorption is slightly dispersed compared with synthetics without absorption.

Layer 1	*Q* *200*	*Z (Depth)* *600*	*Density ρ* *1.9*	*Velocity* *4500*
Layer 2	*50*	*200*	*2.2*	*5000*
Layer 3	*200*	*400*	*2.4*	*3200*
Layer 4	*50*	*200*	*2.3*	*5000*
Layer 5	*200*	*500*	*2.3*	*4500*

Table 1. Five-layer model

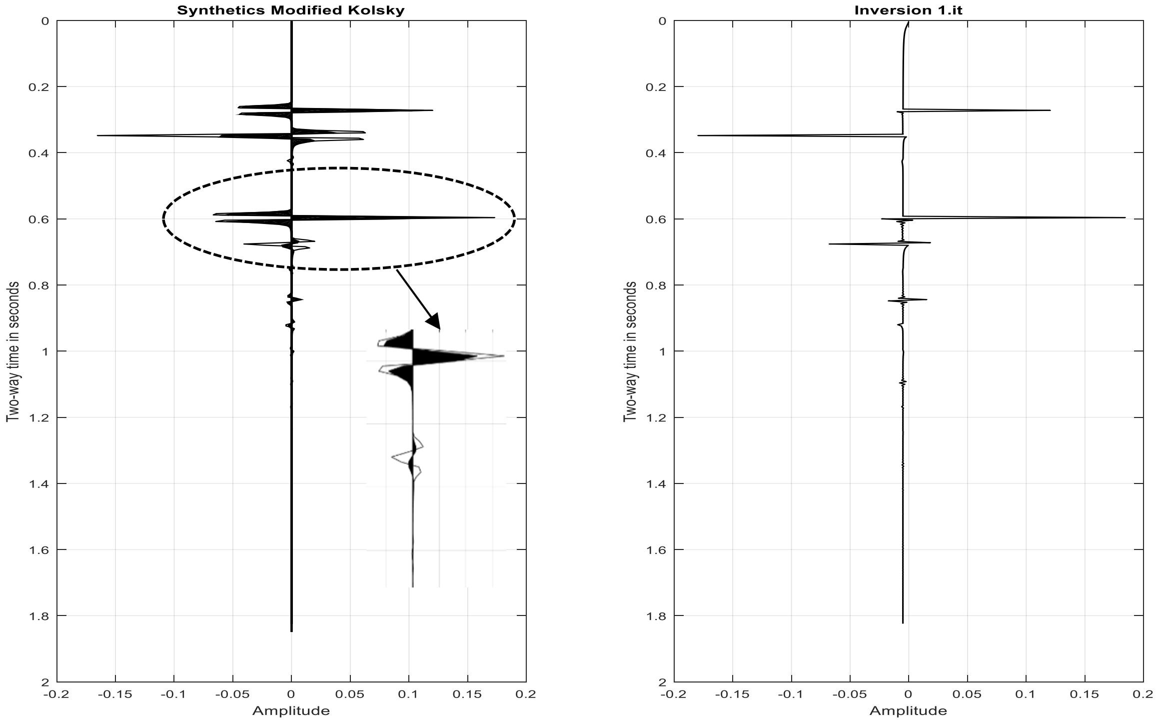

Fig.1.a. From left: Synthetics solution Eq. (18) filled black with absorption, and without absorption. Right: Inversion Eq.28 solution with absorption after 1 iteration.

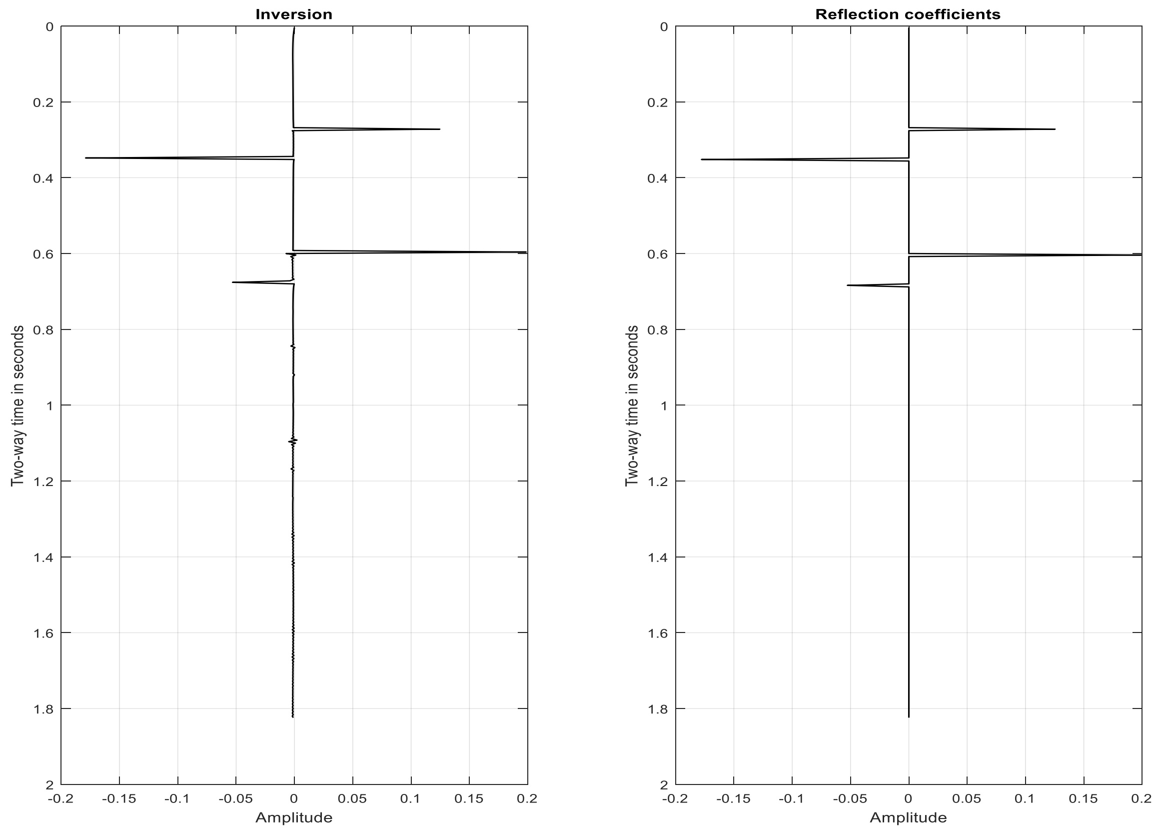

Fig.1.b. From: left, Solution Eq. (28) inversion for the 5.th iteration, Right:Reflection coefficient for the model (Table 1).

For our discussion of the inversion we need to introduce the impedance. The relation between r and the impedance makes it possible to compute the impedance for every solution of r. If the acoustic impedance I_0 is known at z = o, (or at any depth), I is also uniquely determined as a function of τ. From Eq. (8) we can deduce after converting to two-way time:

$$I = I_0 \exp(2\int_0^{\tau} r(\tau)d\tau) \qquad (29)$$

where $I_0=\rho_0 v_0$ and $I= \rho v$. I/I_0 will give us a dimensionless relative impedance that is presented on fig.2.

Now we will plot the impedance of the model and compare it with the impedance of the solution for first and last iteration. Fig.2 shows the full inversion where the impedance model is marked with dotted line.

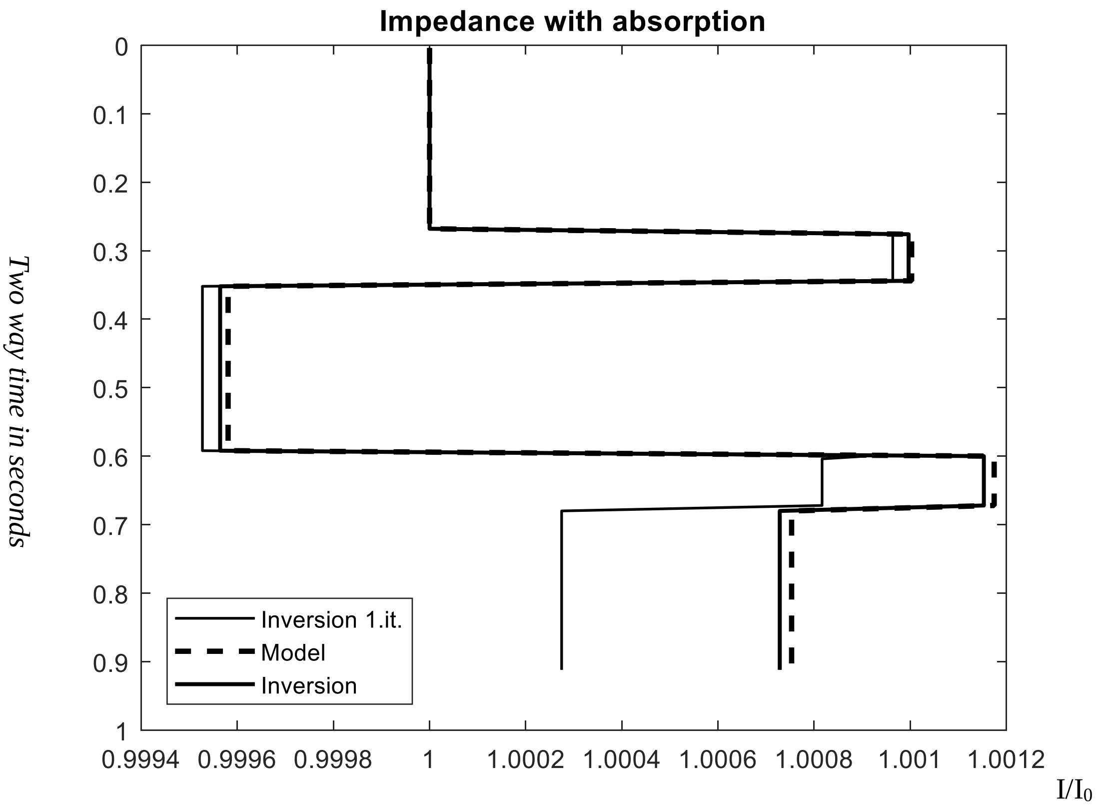

Fig.2. From left: solution Eq.(29) for first and last iteration (inversion) compared with the model (dotted line).Absorption and multiples are removed.

We notice that the interbed multiples introduced in the model by iterations and easy to see on fig.1.a are very well removed in the inversion after 5.iterations (fig.1.b). We also can confirm this on fig.2. A smoother graph on fig.2. suggests that interbed multiples are removed, and a graph close to the model indicates we have restored transmission loss and attenuation.

Inversion using changed Q-values

To see the importance of the correct Q-value in the inversion process, different Q-values are used in the inversion than in the input synthetic seismogram. Fig.3.a. shows over-estimation of damping where the Q-values of the model are scaled with a factor of 0.9 in the inversion giving more absorption in the inversion than in the synthetics. The impedance shows that the inversion does not give as good reconstruction of the model (dotted line) compared to the inversion where the same Q-values where used both for the synthetics and the inversion. (Fig.2). Fig.3.b. shows under-estimation of damping scaling the Q-values with a factor of 1.2. Also this example shows deviation from fig.2.

The impedance on fig.2 compared with fig.3.a and b. show clearly that we can recover the impedance well when the Q-value estimated in the inversion is the same as is used in the model and both over and under-estimation can give deviation from the model.

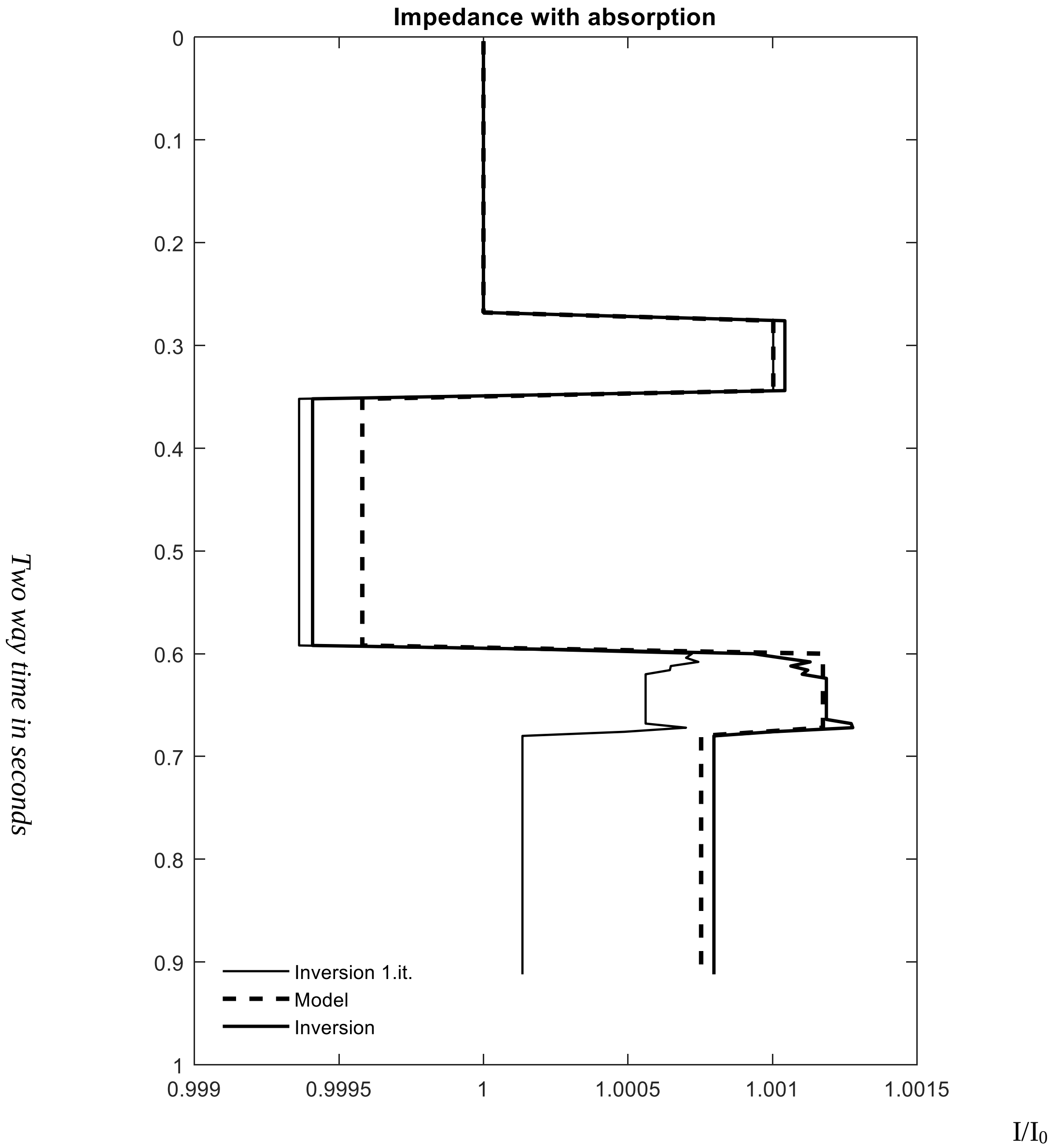

Fig.3.a. Q-values used in solution fig.2 is scaled with a factor 0.9. (over-estimation of absorption)

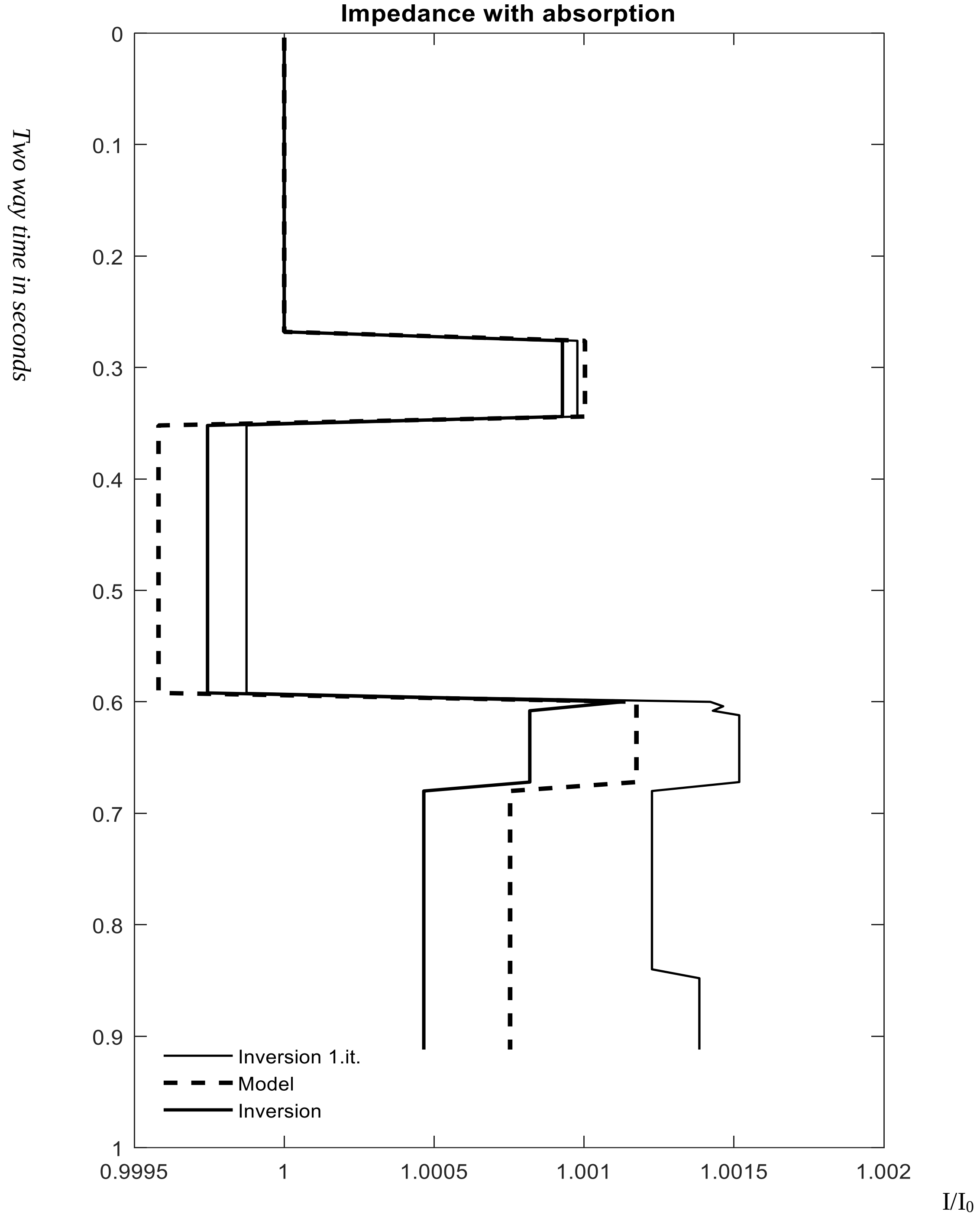

Fig.3.b. Q-values used in solution fig.2 is scaled with a factor 1.2. (under-estimation of absorption)

Inversion with noise

Finally we added noise to the data. The resulting impedance is showed on fig.3.c.. 1. Iteration in inversion is closer to model than the full inversion. However, the interbed multiple after latest arrival are not removed before full inversion.

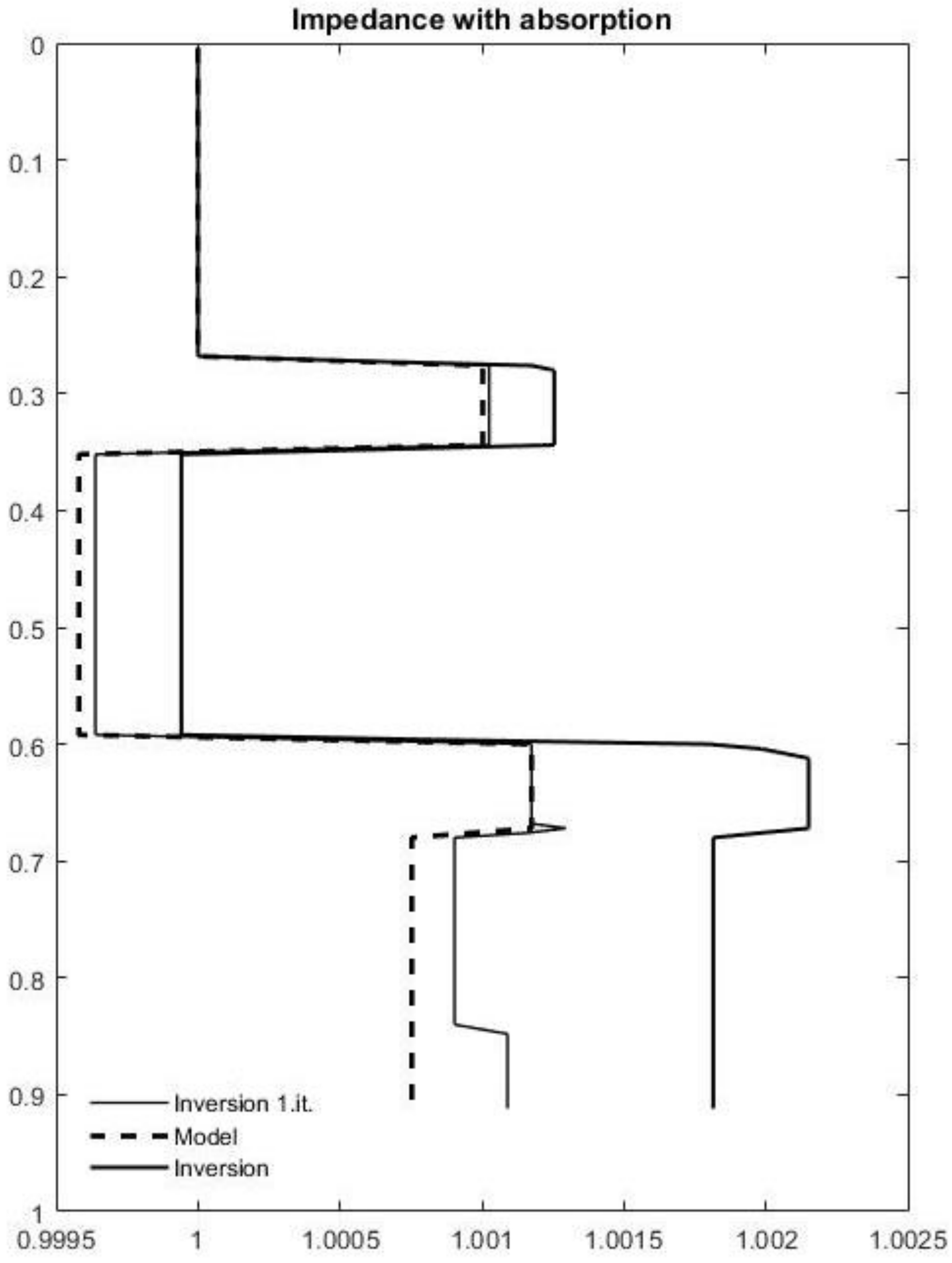

Fig.3.c. Noise is added to the synthetics

Conclusion

The results of the preceding sections show that the Riccati equation with Q-filtering provides a method for the construction of synthetic reflection seismograms that is a continuation of the method introduced by Nilsen and Gjevik. As long as we use the same absorption model (same Q-values) in the inversion than in the input synthetic seismogram we will get a reasonable good result..

Moreover does this theory describe a method for inverting reflection data, i.e. computing the variations in the acoustic impedance within a reflecting layer that can be used in many ways. It corrects phase, compensates frequency loss, removes interbed multiples and compensates transmission loss in one single process. We have been able to see this on plots of impedance variation following the iteration procedure.

So far we have tested the abilities of the inversion method by inverting a synthetic reflection seismograms computed from a simple model. It would, however, be interesting to apply the present inversion method to real reflection data from complex structures. A number of problems will then arise and, in the case of no absorption, some of them were discussed in the paper of Nilsen and Gjevik. (1978).

The main problem is, however, that what Gjevik asked is not fully answered: will one lose so much information or introduce so many errors through the process that the inversion becomes meaningless when applied to real prospecting? In view of the success of the introduction of inverse Q-filtering this could soon be answered and we plan to study these problems in the future.

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